Information Sciences Letters
An International Journal

http://dx.doi.org/10.12785/isl/040207

Analytical Solutions for Free Vibration of Strongly Nonlinear Oscillators

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Received: 11 May 2014, Revised: 24 Apr. 2015, Accepted: 25 Apr. 2015
Published online: 1 May 2015

Abstract: In this paper, a nonlinear free vibration of strongly nonlinear oscillators is studied. To this end, we propose a novel technique using He's frequency-amplitude formulation and He's energy balance method called iteration perturbation procedure. A novel technique called iteration perturbation procedure and variational iteration method are presented to obtain the relationship between amplitude and angular frequency. The obtained results are compared with the numerical solution obtained by using the Runge–Kutta method and shown in graphs indicating the effectiveness and convenience of the analytical approximate solutions. These approaches are very effective and simple and with only one iteration leads to high accuracy of the solutions. It is predicted that those methods can be found wide applications in engineering problems, as indicated in this paper.

Keywords: Periodic solution, nonlinear oscillators, iteration perturbation procedure, variational iteration method.

1 Introduction

Nonlinear phenomena play a crucial role in applied mechanics and physics and have a significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion. Most of real systems are modeled by nonlinear differential equations which are important issues in mechanical structures, mathematical physics and engineering. It is very interesting for the scientific community to use applied mathematics to solve dynamic problems. In recent years, much attention has been devoted to the advanced techniques to investigate the nonlinear systems such as frequency-amplitude formulation (FAF) [1,2,3], energy balance method (EBM) [4,5,6,7], variational iteration method (VIM) [8, 9,10,11], homotopy perturbation method (HPM) [12,13, 14], homotopy analysis method (HAM) [15,16], variational approach (VA) [17,18], max–min approach (MNA) [19] and which are introduced for nonlinear oscillatory systems.

In the following section of this paper, the iteration perturbation procedure and the variational iteration method are applied to an important and interesting problems in nonlinear vibration systems.

2 Methods of solution

2.1 Description of the iteration perturbation procedure

We consider a generalized nonlinear oscillator in the form:

$$\ddot{u}+f(u,\,\dot{u},\,\ddot{u})=0,\;\; u(0)=A,\;\; \dot{u}(0)=0. \tag{1}$$

Introducing the following new function $H(t)$ in the form:

$$H(t)=\int_0^T(\ddot{u}+f(u,\,\dot{u},\,\ddot{u}))\cos(\omega t)dt=0,\;\; T=\frac{2\pi}{\omega}. \tag{2}$$

We begin the procedure with the simplest trial function to determine the angular frequency ω:

$$u=A\cos\omega t. \tag{3}$$

Substituting the above trial functions into Eq. (2) results in, the following residual

$$H(t)=\int_0^{2\pi/\omega}\left[-A\omega^2\cos\omega t+f\left(A\cos\omega t,-A\omega\sin\omega t,\right.\right.$$
$$\left.\left.-A\omega^2\cos\omega t\right)\cos(\omega t)\right]dt=0. \tag{4}$$

Solving the above equation, the relationship between the amplitude and frequency of the oscillator can be obtained:

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2.2 Description of the variational iteration method

To illustrate its basic concepts of the variational iteration method, we consider the following general differential equation [8]:

$$Lu + Nu = g(x), \tag{5}$$

where, L is a linear operator, and N a nonlinear operator, $g(x)$ an inhomogeneous or forcing term. According to the variational iteration method, we can construct a correct functional as follows:

$$u_{(n+1)}(t) = u_n(t) + \int_0^t \lambda \left[Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\right] d\tau, \tag{6}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscript n denotes the *n-th* approximation, $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$.

In the coming sections of the paper, we clarify the rigid rood rocks on the circular surface and the nonlinear free vibration of an oscillator with inertia and static type cubic nonlinearities by using iteration perturbation procedure and variational iteration method. Besides numerical Runge-Kutta method of order 4 will be compared with the analytical results.

3 Motion of a rocking rigid rod

The motion's equation of the rigid rod which rocks on the circular surface without slipping is: (7) where l is rigid rod's length, r is radius of circular surface and u is the function of angle of each time [20,21,22].

The motion equation of the system by applying Lagrangian [20] can be found as:

$$\left(\frac{1}{12}l^2 + r^2u^2\right)\ddot{u} + r^2u\,\dot{u}^2 + r\,g\,u\,\cos u = 0, \tag{7}$$

under the initial conditions:

$$u(0) = A, \quad \dot{u}(0) = 0.$$

3.1 Application of iteration perturbation procedure

As we introduced $H(t)$ in Eq. (2) we have

$$\begin{aligned} H(t) = \int_0^{2\pi/\omega} &\left[\left(\tfrac{1}{12}l^2 + r^2u^2\right)\ddot{u} + r^2u\,\dot{u}^2\right. \\ &\left. + r\,g\,u\,\cos u\right]\cos(\omega t)dt = 0. \end{aligned} \tag{8}$$

Substituting the above trial functions into Eq. (8) results in, the following residual

$$H(t) = \frac{A\pi}{768\omega}\left[(192 - 72A^2 + 5A^4)gr - 16(l^2 + 6A^2r^2)\omega^2\right] = 0. \tag{9}$$

Solving the above equation, the relationship between the amplitude and frequency of the oscillator can be obtained as follow:

$$\omega = \sqrt{\frac{192g\,r - 72A^2g\,r + 5A^4g\,r}{16l^2 + 96A^2r^2}}, \tag{10}$$

Hence, the approximate solution can be readily obtained

$$u(t) = A\cos\left(\sqrt{\frac{192g\,r - 72A^2g\,r + 5A^4g\,r}{16l^2 + 96A^2r^2}}t\right). \tag{11}$$

3.2 Application of variational iteration method

Assume that the angular frequency of Eq (7) is ω, we have the following linearized equation:

$$\ddot{u} + \omega^2u = 0. \tag{12}$$

So we can rewrite Eq. (7) in the form

$$\ddot{u} + \omega^2u + g(u) = 0. \tag{13}$$

where

$$g(u) = r^2u\dot{u}^2 + rgu(1 - \frac{1}{2}u^2 + \frac{1}{24}u^4) - \omega^2u \tag{14}$$

Applying the variational iteration method, the following iterative formula is formed as:

$$\begin{aligned} u_{(n+1)}(t) = u_n(t) + \int_0^t \lambda &\left[\left(\tfrac{1}{12}l^2 + r^2u^2\right)\ddot{u}(\tau)\right. \\ &\left. + \omega^2u_n(\tau) - g(\tau)\right] d\tau \end{aligned} \tag{15}$$

the stationary conditions can be obtained as follows:

$$\left.\begin{aligned} \lambda''(\tau) + \omega^2\lambda(\tau) = 0, \\ \lambda(\tau)|_{\tau=t} = 0, \\ 1 - \lambda'(\tau)|_{\tau=t} = 0. \end{aligned}\right\} \tag{16}$$

The Lagrange multiplier, therefore, can be identified as;

$$\lambda = \frac{1}{\omega}\sin\omega(\tau - t) \tag{17}$$

Substituting the identified multiplier into Eq. (15) results in the following iteration formula:

$$\begin{aligned} u_{(n+1)}(t) = u_n(t) + \tfrac{1}{\omega}\int_0^t \sin\omega(\tau - t) &\left[\left(\tfrac{1}{12}l^2 + r^2u^2\right)\ddot{u}(\tau)\right. \\ &\left. + r^2u\dot{u}^2 + rgu\left(1 - \tfrac{1}{2}u^2 + \tfrac{1}{24}u^4\right)\right] d\tau \end{aligned} \tag{18}$$

Substituting $u_0 = A\cos\omega t$. as a trail function into Eq (7) yields the residual as follows:

$$\begin{aligned} R_0(t) = &\left(\tfrac{-Al^2\omega^2}{12} - \tfrac{3r^2A^3\omega^2}{4} + \tfrac{r^2A^3\omega^2}{2} - \tfrac{r^2A^3\omega^2}{4}\right. \\ &\left. + rgA - \tfrac{3rgA^3}{8} + \tfrac{5rgA^5}{192}\right)\cos\omega t - \left(\tfrac{r^2A^3\omega^2}{4}\right. \\ &\left. + \tfrac{r^2A^3\omega^2}{8} + \tfrac{rgA^3}{8} - \tfrac{5rgA^5}{384}\right)\cos 3\omega t \\ &+ \tfrac{rgA^5}{384}\cos 5\omega t. \end{aligned} \tag{19}$$

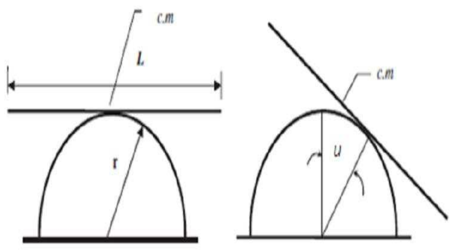

Fig. 1: Schematic of the rigid rod rocks on a circular surface without slipping.

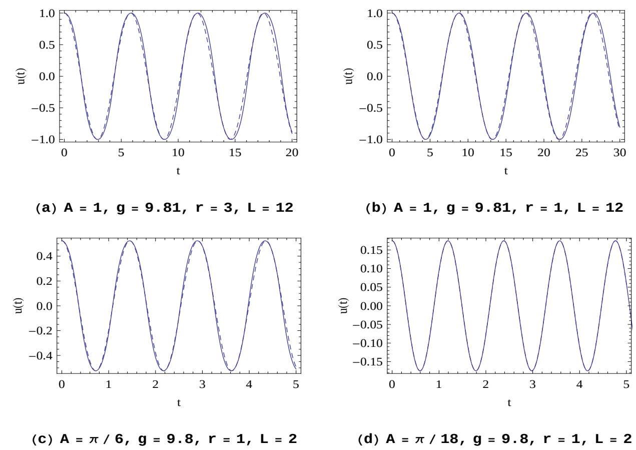

Fig. 2: The comparison between analytical approximate solutions (- - -) and numerical solution (—).

By the formulation (18), we can obtain

$$u_1(t) = A\cos\omega t + \int_0^t \frac{1}{\omega}\sin\omega\,(\tau - t)\,R_0(\tau)\,d\tau \tag{20}$$

In order to ensure that no secular terms appear in u_1, resonance must be avoided. To do so, the coefficient of $cos(\omega t)$ in Eq. (19) requires being zero, i.e.,

$$\omega = \sqrt{\frac{192g\,r - 72A^2 g\,r + 5A^4 g\,r}{16l^2 + 96A^2 r^2}}. \tag{21}$$

Hence, the approximate solution can be readily obtained

$$u(t) = A\cos\left(\sqrt{\frac{192g\,r - 72A^2 g\,r + 5A^4 g\,r}{16l^2 + 96A^2 r^2}}t\right). \tag{22}$$

To illustrate the validity of the analytical approximate solutions for this example, the results are compared with the numerical solution, using fourth order Runge-Kutta method (R-K) in Fig 2 (a-d).

4 Nonlinear free vibration of systems with inertia and static type cubic nonlinearities

Consider free vibration of a conservative, single-degree-of-freedom system with a mass attached to linear and nonlinear springs in series as shown in Fig 3. After transformation, the motion is governed by a nonlinear differential equation of motion [23,24,25,26] as:

$$\left(1 + 3\varepsilon\eta u^2\right)\ddot{u} + 6\varepsilon\eta u\,\dot{u}^2 + \omega_0^2 u + \varepsilon\,\omega_0^2\,u^3 = 0, \tag{23}$$

under the initial conditions:

$$u(0) = A, \quad \dot{u}\,(0) = 0.$$

where

$$\left.\begin{aligned} &u(t) = y_2(t) - y_1(t), \quad \varepsilon = \beta/k_2, \quad \xi = k_2/k_1, \\ &\eta = \tfrac{\xi}{1+\xi}, \qquad \omega_0^2 = \tfrac{k_2}{m(1+\xi)}. \end{aligned}\right\}$$

In which k_1 and k_2 are the linear and nonlinear spring constant, respectively [27]. Parameters ε, β, ν, ω_0, m and ξ are perturbation parameter, coefficient of the nonlinear spring force, deflection of the nonlinear spring, natural frequency, mass and the ratio of the springs constant.

4.1 Application of iteration perturbation procedure

Similar to previous example we have

$$H(t) = \frac{A\pi}{4\omega}\left[\omega_0^2\left(4 + 3\varepsilon A^2\right) - \left(4 + 3\varepsilon\eta A^2\right)\right)\omega^2\right] = 0. \tag{24}$$

Finally, the frequency amplitude relationship can be obtained as:

$$\omega = \sqrt{\frac{4\omega_0^2 + 3\varepsilon\omega_0^2 A^2}{4 + 3\varepsilon\eta A^2}}, \tag{25}$$

Hence, the approximate solution can be readily obtained

$$u(t) = A\cos\left(\sqrt{\frac{4\omega_0^2 + 3\varepsilon\omega_0^2 A^2}{4 + 3\varepsilon\eta A^2}}t\right). \tag{26}$$

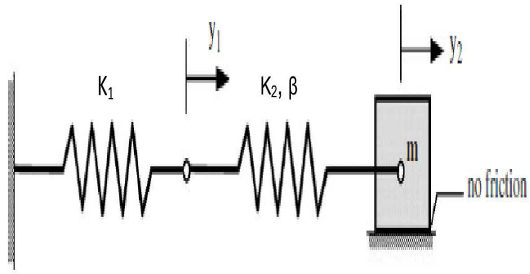

Fig. 3: Nonlinear free vibration of a system of mass with serial linear and nonlinear stiffness on a frictionless contact surface.

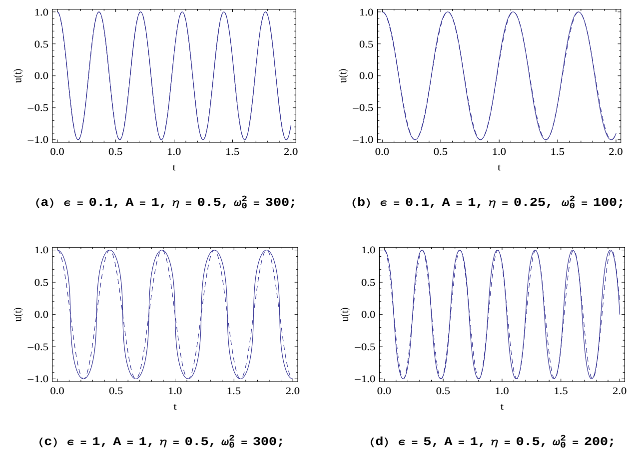

Fig. 4: The comparison between analytical approximate solutions (- - -) and numerical solution (—).

4.2 Application of variational iteration method

According to the previous example, the frequency based on variational iteration method may be expressed as

$$\omega = \sqrt{\frac{4\omega_0^2 + 3\varepsilon\omega_0^2 A^2}{4 + 3\varepsilon\eta A^2}}. \tag{27}$$

Hence, the approximate solution can be readily obtained

$$u(t) = A\cos\left(\sqrt{\frac{4\omega_0^2 + 3\varepsilon\omega_0^2 A^2}{4 + 3\varepsilon\eta A^2}}t\right). \tag{28}$$

To illustrate the validity of the analytical approximate solutions for this example, the results are compared with an accurate numerical solution, using fourth order Runge-Kutta method (R-K) in Fig 4 (a-d).

5 Conclusion

In this paper, we have shown the effectiveness and efficiency of the iteration perturbation procedure introduced in this paper and the variational iteration method in obtaining analytic approximate solutions to nonlinear free vibration of strongly nonlinear oscillators. We compared our results with the exact result obtained numerically by the use of Runge-Kutta fourth-order method and our comparison shows that the two methods considered in this paper give accurate results. Moreover, the two methods showcased in this paper, are very easy and simple to handle as they do not involve rigorous calculation processes as well as complex mathematical ideas. Though more research is required in the light of gaining more information as to how these approximate methods affects real physical systems, this paper presents a step towards a successful and positive implementation of the two methods.

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